Finite length Solenoid potential and field

The origin document was written in 2011. There are sign error and factor 2 error at that time. I update and redo the calculation in 2021. It is a supplementary document for the webpage [https://nukephysik101.wordpress.com/2011/07/17/the-magnetic-field-of-a-finite-length-solenoid/](https://nukephysik101.wordpress.com/2011/07/17/the-magnetic-field-of-a-finite-length-solenoid/). People should also refer to the calculation of the single coil. At the time of writing this document, I was a PhD student at Tokyo University.

Tsz Leung Tang, Saturday, July 10, 2021

The surface current density is (Jackson, 1998):

\[
\vec{K} = \frac{I}{L} \delta(\rho - a)(-\sin \phi, \cos \phi, 0), \quad z \in \left(-\frac{L}{2}, \frac{L}{2}\right).
\]

In above, \(L\) is the total length of the solenoid, \(I\) is the current, \((\rho, \phi, z)\) are the cylindrical coordinate, and \(a\) is the radius of the solenoid. The general form of vector potential is

\[
\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3 x'.
\]

By symmetry, the vector potential is azimuthal.

\[
\vec{A} = A_{\phi} \hat{\phi} = A_{\phi} \int_{L}^{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{0}^{2\pi} \frac{\delta(\rho' - a) \cos \phi'}{\sqrt{\rho^2 + \rho'^2 + (z - z')^2 - 2a\rho \cos \phi'}} \rho' d\rho' d\phi' dz',
\]

\[
A_{\phi} = \frac{\mu_0 I a}{2\pi L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{0}^{2\pi} \frac{\cos \phi'}{\sqrt{\rho^2 + a^2 + (z - z')^2 - 2a\rho \cos \phi'}} d\phi' dz'.
\]

Simplify the form by setting \(\zeta = (z - z')\) and the integration of \(\zeta\) is a log function (Edmund E. Callaghan, 1960), we have,

\[
\int_{0}^{\pi} \cos \phi' \left[ \ln \left( \zeta + \sqrt{\zeta^2 + \rho^2 + a^2 - 2a\rho \cos \phi'} \right) \right]^{\zeta_+} \left[ \ln \left( \zeta - \sqrt{\zeta^2 + \rho^2 + a^2 - 2a\rho \cos \phi'} \right) \right]^{\zeta_-} d\phi' = \left[ \int_{0}^{\pi} \cos \phi' \ln(\zeta + a(\zeta)) \right]^{\zeta_+}_{\zeta_-},
\]

\[
\alpha(\zeta) = \sqrt{\zeta^2 + \rho^2 + a^2 - 2a\rho \cos \phi'}, \quad \zeta_\pm = z \mp \frac{L}{2}.
\]

Then,

\[
A_{\phi} = \frac{\mu_0 I a}{2\pi L} \left[ \int_{0}^{\pi} \cos \phi' \ln(\zeta + a(\zeta)) \right]^{\zeta_+}_{\zeta_-}
\]

Integration by path gives,
\[
\int_{0}^{\pi} \cos \phi' \ln(\zeta + \alpha(\zeta)) \, d\phi' = \sin \phi' \ln(\zeta + \alpha(\zeta)) \bigg|_{0}^{2\pi} - \int_{0}^{\pi} \sin \phi' \, d(\ln(\zeta + \alpha(\zeta)))
\]

The first term is zero because of \(\sin \phi'\), and the derivative of \(\ln(\zeta + \alpha(\zeta))\) is:

\[
\frac{d \ln(\zeta + \alpha(\zeta))}{d\phi'} = \frac{\rho_{a} \sin \phi'}{(\alpha(\zeta) + \zeta)\alpha(\zeta)}
\]

Multiple by \((\alpha(\zeta) - \zeta)/(\alpha(\zeta) - \zeta)\)

\[
\frac{\alpha(\zeta) - \zeta}{\rho_{a} \sin \phi'} \frac{\rho_{a} \sin \phi'}{(\alpha^{2}(\zeta) - \zeta^{2})\alpha(\zeta)} = \frac{\zeta \rho_{a} \sin \phi'}{(\rho^{2} + a^{2} - 2\rho_{a} \cos \phi') \alpha(\zeta)} - \frac{\zeta \rho_{a} \sin \phi'}{(\rho^{2} + a^{2} - 2\rho_{a} \cos \phi') \alpha(\zeta)}
\]

The first term is a constant of \(\zeta\), and \([x]_{\zeta}^{\zeta+} = 0\), then,

\[
\int_{0}^{\pi} \sin \phi' \, d(\ln(\zeta + \alpha(\zeta))) = - \int_{0}^{\pi} \frac{\zeta \rho_{a} \sin^{2} \phi'}{(\rho^{2} + a^{2} - 2\rho_{a} \cos \phi') \alpha(\zeta)} \, d\phi'
\]

\[
A_{\phi} = \frac{\mu_{0} 1 a^2 \rho}{2\pi L} \left[ \int_{0}^{\pi} \frac{\sin^{2} \phi'}{(\rho^{2} + a^{2} - 2\rho_{a} \cos \phi') \sqrt{\zeta^{2} + \rho^{2} + a^{2} - 2\rho_{a} \cos \phi'}} \, d\phi' \right]_{\zeta-}^{\zeta+}
\]

Now, since the \(\cos \phi\) is the same as interval \((-\pi, 0)\) and \((0, \pi)\), we can change the sign, and replacing \(\phi = 2\theta\), using \(\cos(2\theta) = 1 - 2\sin^2 \theta\), we have

\[
\int_{0}^{\pi} \frac{\sin^2 \phi'}{(\rho^2 + a^2 - 2\rho a \cos \phi') \sqrt{\zeta^2 + \rho^2 + a^2 - 2\rho a \cos \phi'}} \, d\phi'
\]

\[
= \int_{0}^{\pi} \sin^2 \phi' \, d\phi'
\]

\[
= 2 \int_{0}^{\pi/2} \sin^2(2\theta) \sqrt{((a + \rho)^2 - 4\rho a \sin^2 \theta) \zeta^2 + (a + \rho)^2 - 4\rho a \sin^2 \theta} \, d\theta
\]

\[
= \frac{k h^2}{4(\sqrt{a^2 \rho})^3} \int_{0}^{\pi/2} \frac{\sin^2(2\theta)}{(1 - h^2 \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}} \, d\theta
\]

\[
= \frac{k h^2}{4(\sqrt{a^2 \rho})^3} \int_{0}^{\pi/2} \frac{\sin^2 \theta - \sin^4 \theta}{(1 - h^2 \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}} \, d\theta, \quad \sin^2(2\theta) = 4\sin^2 \theta - 4\sin^4 \theta
\]

\[
h^2 = \frac{4a \rho}{(a + \rho)^2}
\]

\[
k^2 = \frac{4a \rho}{(a + \rho)^2 + \zeta^2}
\]

In the above calculation, in the 1st step, the \(\cos \phi\) denominator change sign means the while function flipped horizontally on the \(\phi = \frac{\pi}{2}\).
The integral can be split into 2 parts, the first part is (Milton Abramowitz, 1965) (NIST Digital Library of Mathematical Functions):

\[
\int_{0}^{\pi} \frac{\sin^2 \theta}{(1 - h^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} d\theta = -\frac{1}{h^2} \int_{0}^{\pi} \frac{1}{(1 - h^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} d\theta
\]

\[
= \frac{1}{h^2} \int_{0}^{\pi} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} - \frac{1}{(1 - h^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} d\theta
\]

\[
= \frac{1}{h^2} \left( \Pi(h^2, k^2) - K(k^2) \right)
\]

Here

\[
\Pi(n, m) = \int_{0}^{\pi} \frac{1}{(1 - n \sin^2 \theta)\sqrt{1 - m \sin^2 \theta}} d\theta
\]

It is the elliptic integral of 3rd kind.

The 2nd part is:

\[
\int_{0}^{\pi} \frac{-\sin^4 \theta}{(1 - h^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} d\theta = \frac{1}{h^4} \int_{0}^{\pi} \frac{1 - h^4 \sin^4 \theta - 1}{(1 - h^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} d\theta
\]

\[
= \frac{1}{h^4} \int_{0}^{\pi} \frac{1 + h^2 \sin^2 \theta}{\sqrt{1 - k^2 \sin^2 \theta}} - \frac{1}{(1 - h^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} d\theta
\]

\[
= \frac{1}{h^4} \left( K(k^2) + \frac{h^2}{k^2} \left( K(k^2) - E(k^2) \right) - \Pi(h^2, k^2) \right)
\]

Thus, combine everything and we have,

\[
A_\phi = \frac{\mu_0 I}{2\pi L} \left[ a \zeta k \left( \frac{k^2 + h^2 - h^2 k^2}{h^2 k^2} K(k^2) - \frac{1}{k^2} E(k^2) + \frac{h^2 - 1}{h^2} \Pi(h^2, k^2) \right) \right]_{\zeta^+}^{\zeta^-}
\]

The magnetic field is the curl

\[
B_\rho = [\nabla \times A_\phi]_\rho = -\frac{\partial}{\partial z} (A_\phi) = -\frac{\partial}{\partial \zeta} (A_\phi)
\]

\[
B_z = [\nabla \times A_\phi]_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\phi) = \frac{1}{\rho} A_\phi + \frac{\partial A_\phi}{\partial \rho}
\]

Using the derivative formulae for elliptic integral:
\[
\frac{d}{dk} K(k^2) = -\frac{1}{k} K(k^2) + \frac{1}{k(1-k^2)} E(k^2)
\]
\[
\frac{d}{dk} E(k^2) = -\frac{1}{k} K(k^2) + \frac{1}{k} E(k^2)
\]
\[
\frac{d}{dk} \Pi(h^2, k^2) = -\frac{k}{(1-k^2)(h^2-k^2)} E(k^2) - \frac{k}{(h^2-k^2)} \Pi(v^2, x^2)
\]

Since derivative of \( \zeta \) and \( \rho \) is through derivative of \( k \), we compute,
\[
\frac{d}{dk} \left( k \left( \frac{k^2 + h^2 - h^2 k^2}{h^2 k^2} K(k^2) - \frac{1}{k^2} E(k^2) \right) \right)
\]
\[
= -\frac{1}{k^2} K(k^2) + \frac{h^2}{k^2(h^2-k^2)} E(k^2) + \frac{h^2 - 1}{(h^2-k^2)} \Pi(h^2, k^2)
\]

And,
\[
\frac{dk}{dz} = -\frac{k^3 \left( z + \frac{L}{2} \right)}{4\alpha \rho}, \quad \frac{dk}{d\rho} = \frac{k}{2\rho} - \frac{k^3 (a + \rho)}{4\alpha \rho}
\]
\[
\frac{d}{dz} (zf(z)) = f(z) + z \frac{df(z)}{dz}, \quad \frac{d}{d\rho} \left( \frac{1}{\sqrt{\rho}} f(\rho) \right) = -\frac{f(\rho)}{2\sqrt{\rho^3}} + \frac{1}{\sqrt{\rho}} \frac{df(\rho)}{d\rho}
\]

Then
\[
B_\rho = \frac{\mu_0}{2\pi L} \sqrt{\frac{a}{\rho}} \left[ \left( \frac{k^2 - 2}{k} K(k^2) + \frac{2}{k} E(k^2) \right) \right]_{\zeta_+}^{\zeta_-}
\]

Or, by using integration identity
\[
\frac{d}{dx} \int_a^b f(x) dx = f(b) - f(a)
\]

And the original formula of \( A_\phi \)
\[
A_\phi = \frac{\mu_0}{2\pi L} \int_{\frac{L}{2}}^{\frac{L}{2}} \frac{\cos \phi'}{\sqrt{\rho^2 + a^2 + (z-z')^2 - 2\alpha \rho \cos \phi'}} d\phi' dz'.
\]

We have,
\[
B_\rho = \frac{\mu_0}{2\pi L} \left[ \left( \frac{1}{\sqrt{\xi^2 + \rho^2 + a^2 - 2\alpha \rho \cos \phi'}} \right)_{\xi_+}^{\xi_-} \right]
\]

Then, we can verify the two expression is the same. Our result for \( B_\rho \) is
\[ B_p = \frac{\mu_0 I}{2\pi L} \sqrt{\frac{\alpha}{\rho}} \left[ \left( \frac{k^2 - 2}{k} K(k^2) + \frac{2}{k} E(k^2) \right) \right]_{\xi}^{\zeta}, \quad k^2 = \frac{4a\rho}{(a + \rho)^2 + \zeta^2} \]

To compute the z component, we need to be careful the \( \Pi(h^2, k^2) \), since \( h^2 \) also contains \( \rho \). And the z-component is:

\[ B_z = \frac{\mu_0 I}{2\pi L} \left( \frac{1}{2\sqrt{a\rho}} \left[ \zeta k \left( K(k^2) + \frac{\alpha - \rho}{\alpha + \rho} \Pi(h^2, k^2) \right) \right]_{\xi}^{\zeta} \right) \]

To verify the \( B_z \), we can compute \( \frac{\partial A_\phi}{\partial \rho} \) from

\[ A_\phi = \frac{\mu_0 Ia}{2\pi L} \left[ \int_0^\pi \cos \phi' \ln(\zeta + \alpha(\zeta)) \, d\phi' \right]_{\xi}^{\zeta} \]

By

\[ \frac{\partial}{\partial \rho} \ln(\zeta + \alpha(\zeta)) = \frac{\rho - a \cos \phi'}{\alpha(\alpha(\zeta) + \zeta)} \]

Using the same trick

\[ \frac{\rho - a \cos \phi'}{\alpha(\alpha(\zeta) + \zeta)} = \frac{(\rho - a \cos \phi')(\alpha(\zeta) - \zeta)}{\alpha(\zeta)^2(\zeta + \zeta)} \]

\[ = \frac{(\rho - a \cos \phi'\alpha(\zeta) - \zeta)}{\alpha(\zeta)(\rho^2 + a^2 - 2\rho a \cos \phi')} = \frac{(\rho - a \cos \phi') - (\rho - a \cos \phi')\zeta}{\alpha(\zeta)(\rho^2 + a^2 - 2\rho a \cos \phi')} \]

Therefore:

\[ \frac{\partial A_\phi}{\partial \rho} = \frac{\mu_0 Ia}{2\pi L} \left[ \int_0^\pi \frac{-\zeta \rho \cos \phi' + a \zeta \cos^2 \phi'}{\alpha(\zeta)(\rho^2 + a^2 - 2\rho a \cos \phi')} \, d\phi' \right]_{\xi}^{\zeta} \]

Combined with

\[ \frac{1}{\rho} A_\phi = \frac{\mu_0 Ia}{2\pi L} \left[ \int_0^\pi \frac{a \zeta \sin^2 \phi'}{\alpha(\zeta)(\rho^2 + a^2 - 2\rho a \cos \phi')} \, d\phi' \right]_{\xi}^{\zeta} \]

Then the magnetic field is

\[ B_z = \frac{\mu_0 Ia}{2\pi L} \left[ \int_0^\pi \frac{a - \rho \cos \phi'}{\alpha(\zeta)(\rho^2 + a^2 - 2\rho a \cos \phi')} \, d\phi' \right]_{\xi}^{\zeta} \]

With the same trick,

\[ \int_0^\pi \frac{a - \rho \cos \phi'}{(\rho^2 + a^2 - 2\rho a \cos \phi')\sqrt{\xi^2 + \rho^2 + a^2 - 2\rho a \cos \phi'}} \, d\phi' \]
\[
\int_0^\pi \frac{a + \rho \cos \phi'}{\sqrt{\xi^2 + \rho^2 + a^2 + 2\rho a \cos \phi'}} \, d\phi' = 2 \int_0^{\pi/2} \frac{a + \rho \cos(2\theta)}{((a + \rho)^2 - 4\rho a \sin^2 \theta)\sqrt{\xi^2 + (a + \rho)^2 - 4\rho a \sin^2 \theta}} \, d\theta
\]

\[
= \frac{kh^2}{4(\sqrt{a\rho})^3} \int_0^{\pi/2} (a + \rho - 2\rho \sin^2 \theta) \sqrt{1 - (1 - k^2 \sin^2 \theta)} \, d\theta
\]

\[
= \frac{k}{4(\sqrt{a\rho})^3} \left( (h^2(a + \rho) - 2\rho)\Pi(h^2, k^2) + 2\rho K(k^2) \right)
\]

\[
= \frac{k}{2a\sqrt{a\rho}} \left( K(k^2) + \frac{(a - \rho)}{a + \rho} \Pi(h^2, k^2) \right)
\]

Thus we get the same result.

\[
B_z = \frac{\mu_0 I}{2\pi L} \frac{1}{2\sqrt{a\rho}} \left[ \zeta k \left( K(k^2) + \frac{a - \rho}{a + \rho} \Pi(h^2, k^2) \right) \right]_{\zeta^-}^{\zeta^+}
\]

In conclusion, the field is defined by:

\[
A_\phi = \frac{\mu_0 I}{2\pi L} \sqrt{\frac{a}{\rho}} \left[ \zeta k \left( \frac{k^2 + h^2 - h^2 k^2}{h^2 k^2} K(k^2) - \frac{1}{k^2} E(k^2) + \frac{h^2 - 1}{h^2} \Pi(h^2, k^2) \right) \right]_{\zeta^-}^{\zeta^+}
\]

\[
B_\rho = \frac{\mu_0 I}{2\pi L} \frac{a}{\sqrt{\rho}} \left[ \left( \frac{k^2 - 2}{k} K(k^2) + \frac{2}{k} E(k^2) \right) \right]_{\zeta^-}^{\zeta^+}
\]

\[
B_z = \frac{\mu_0 I}{2\pi L} \frac{1}{2\sqrt{a\rho}} \left[ \zeta k \left( K(k^2) + \frac{a - \rho}{a + \rho} \Pi(h^2, k^2) \right) \right]_{\zeta^-}^{\zeta^+}
\]

With

\[h^2 = \frac{4a\rho}{(a + \rho)^2}, \quad k^2 = \frac{4a\rho}{(a + \rho)^2 + \zeta^2}, \quad \zeta = z \pm \frac{L}{2}\]

Here are some plots:
Works Cited


